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Jun-Lin Liu

T.J. Sager

Bruce M. McMillin

*Missouri University of Science and Technology, ff@mst.edu*

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# AN IMPROVED CHARACTERIZATION OF 1-STEP RECOVERABLE EMBEDDINGS: RINGS IN HYPERCUBES<sup>†</sup>

Jun-Lin Liu<sup>‡</sup>,

Computer & Comm. Research Lab.  
Industrial Technology Research Institute  
v400, 195-11 Sec. 4 Chung Hsing Rd.  
Chutung, Taiwan 310, R.O.C

Thomas J. Sager, and Bruce M. McMillin  
Department of Computer Science  
University of Missouri at Rolla  
Rolla, Missouri 65401  
ff@cs.umr.edu

**Abstract** - An embedding is 1-step recoverable if any single fault occurs, the embedding can be reconfigured in one reconfiguration step to maintain the structure of the embedded graph. In this paper we present an efficient scheme to construct this type of 1-step recoverable ring embeddings in the hypercube. Our scheme will guarantee finding a 1-step recoverable embedding of a length- $k$  (even) ring in a  $d$ -cube where  $6 \leq k \leq (3/4)2^d$  and  $d \geq 3$ , provided such an embedding exists. Unlike previously proposed schemes, we solve the general problem of embedding rings of different lengths, and the resulting embeddings are of smaller expansion than in previous proposals. A sufficient condition for the non-existence of 1-step recoverable embeddings of rings of length  $> (3/4)2^d$  in  $d$ -cubes is also given.

**Keywords:** Ring, Hypercube, Embedding, Fault Tolerance, Reconfiguration.

## I. Introduction

The hypercube has been considered as a useful host topology to simulate many application graphs [1,2,3,6,12,13,14], and most research assumes that the hypercube is fault-free. However, because of increasingly large parallel computer architectures, a processor failure during a long computation is not uncommon, and hours of computation will be wasted unless the system can be properly reconfigured to continue normal operation. Several reconfigurable embedding schemes have been given for a fixed hypercube (without redundant processors and links) in recent years [4,5,7,8,9,10,11]. The fundamental idea for these schemes is to embed an application graph on the hypercube without fully utilizing the nodes on the hypercube. The remaining nodes can be used as spare to reconfigure the embedding in case of faults. The result of this research shows that by carefully embedding the application graphs, the topological properties of the embedding can be preserved under fault conditions, and reconfiguration can be carried out efficiently.

We investigated the designs for reconfigurable embeddings of the ring in the hypercube with the aim of minimizing reconfiguration cost and performance degradation. The cost is measured by the number of node-state changes or reconfiguration steps [15] needed for processing of the reconfiguration, and the perfor-

mance degradation is characterized as the dilation of the new embedding after reconfiguration. Proposals for constructing such embeddings have been given [4,5,8,9,11] by others and us. However, the area of designing 1-step recoverable embeddings has not received much attention; only [5,8,11] address or partially address this problem. An embedding is 1-step recoverable if any single fault occurs the embedding can be reconfigured in one reconfiguration step such that the new embedding is still a ring of the original length. 1-step recoverable embeddings are most interesting because of their efficient reconfigurability due to faults. For convenience, we will use the term "1SRE" to denote the phrase "1-step recoverable embedding" throughout the paper.

The existing schemes for constructing 1SRE's are either restricted ([5,11]), applicable only to rings of some particular lengths, or result in embeddings with large expansion ([8]) (a low percentage of processor utilization on the hypercube). Thus, in this paper, we propose an efficient scheme to systematically construct a 1SRE of a length- $k$  ring in a  $d$ -cube where  $k$  is even,  $6 \leq k \leq (3/4)2^d$ , and  $d \geq 3$ . Our scheme is based on a composition idea by which a 1SRE in a given dimension hypercube is formed by combining two 1SRE's in two lower dimension hypercubes. The running time complexity of our scheme is linear to the length of the ring embedded, and there are many more rings of different lengths which are applicable to our scheme than those to [5,11]. Also, the resulting 1SRE's of our scheme are of smaller expansion than that of [8].

In Section II, we summarize the failure model and reconfiguration algorithm reported in [4,15]. The details of our proposed scheme and proofs for the correctness of the scheme are given in Section III. In Section IV, we show the complexity of our scheme and compare the scheme to other existing ones. A sufficient condition for the non-existence of 1SRE's of rings of length  $> (3/4)2^d$  in a  $d$ -cube is given in Section V. The concluding remarks and open problems follow as Section VI.

## II. Failure Model and Reconfiguration Algorithm

The failure model and reconfiguration algorithm used in this paper follow those defined in [4,15]. We will briefly summarize their ideas and approaches here in the interest of completeness.

### Failure Model

For an embedding of a length- $k$  ring in a  $d$ -cube, every node on the  $d$ -cube is assigned either an active state or a spare state, and an active state is denoted as a positive integer from 1 through  $k$ , and a spare state is denoted as 0. A node is an active node (a spare node) if it has an active state (a spare state). An example of state assignments is shown in Fig. 1, where a length-6 ring and its embedding in a 3-cube is given in Fig. 1(a) and 1(b), respectively. Throughout the figures in this paper, we will use directed arcs in the hypercube to denote an embedding.

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<sup>‡</sup> This work was performed while at the University of Missouri-Rolla.

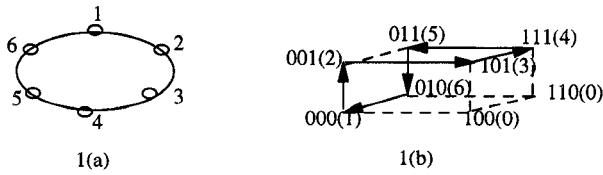


Fig. 1. (a) A length-6 ring. (b) An embedding of a length-6 ring in a 3-cube with every node labeled with its state in parentheses.

When a node on the hypercube becomes faulty, its state is changed to the faulty state, denoted as -1. For example, if the node 011 in Fig. 1(b) becomes faulty, its state will be changed from 5 to -1. Thus, fault detection on an embedding will be based on the identification of a missing state of the  $k$  possible active states. It is assumed that reliable fault diagnosis mechanisms are available and if a node becomes faulty its state changes to -1. In this paper we deal with single-fault scenarios only.

#### Reconfiguration Algorithm

Let  $a_i$  be a node with active state  $i$ . For an embedding of a length- $k$  ring in a  $d$ -cube,  $a_i$  maintains a backup of the work environment of  $a_{i+1}$ , and is responsible for detecting the fault and reconfiguring the embedding due to the missing of active state  $t+1$ . When  $a_{t+1}$  becomes faulty (state changed from  $t+1$  to -1),  $a_i$  will detect this fault and act as a local supervisor to invoke the following reconfiguration actions:

- 1) Compute  $s = \text{XOR}(a_i, a_{i+1}, a_{i+2})$  (i.e., XOR denotes the bit-wise exclusive-or operation).
- 2) Let the previous state of node  $s$  be  $w$ . Change the state of node  $s$  from  $w$  to  $t+1$ , and assign the work environment of  $a_{t+1}$  to  $s$ .
- 2a)  $w = 0$ . The reconfiguration is done.
- 2b)  $w > 0$ . A propagated fault for recovering active state  $w$  is issued and the reconfiguration processing continues.

Note that in the step 1) above, the four nodes,  $a_i$ ,  $a_{i+1}$ ,  $a_{i+2}$ , and node  $s$  constitute the four nodes of a 2-D plane of the hypercube.

The reconfiguration actions above form the *xor-reconfiguration* algorithm. An example for running the algorithm is shown in Fig. 2. Fig. 2(a) depicts a fault-free embedding of a length-10 ring in a 4-cube. A fault in node 0110 causes the state to change from 5 to -1 (Fig. 2(b)). Since node 0111 with state 4 is responsible for detecting the missing of state 5, it will eventually detect this fault and act as a local supervisor to reconfigure the embedding. With  $\text{XOR}(0111, 0110, 1110) = 1111$ , the state of node 1111 will be changed from 7 to 5, and the work environment of node 0110 will be assigned to node 1111 (Fig. 2(c)). Since the previous state of node 1111 (7) is  $> 0$ , a propagated fault for recovering state 7 is issued and the same reconfiguration actions are repeated. Node 1110 with state 6 then detecting the missing of state 7. With  $\text{XOR}(1110, 1111, 1101) = 1100$ , the spare node 1100 becomes active with state 7, and it is assigned with the old work environment of node 1111 (Fig. 2(d)).

The *cost* of reconfiguring an embedding due to a fault is measured by the number of state changes in the fault-free nodes on the hypercube (i.e., the number of nodes with state changing from spare to active or active to spare). For the example cited, the final reconfiguration causes two state changes: the state of node 1111 changes from state 7 to state 5 and the state of node 1100 changes from 0 to 7.

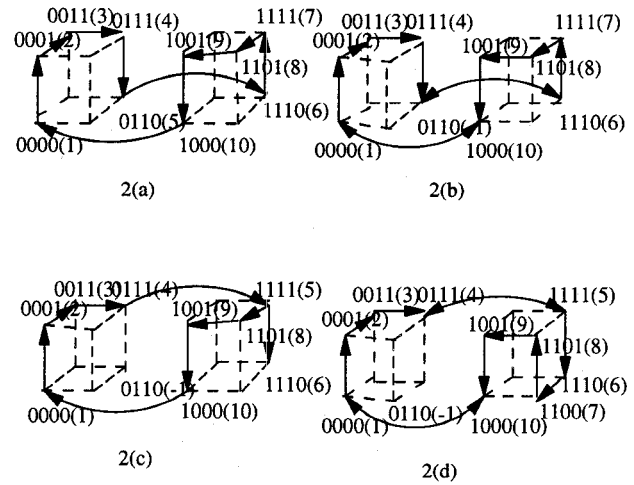


Fig. 2. (a) The fault-free embedding of a length-10 ring in a 4-cube with the fault in node 0110 causing the loss of state 5 (b). (c) The state of a local supervisor, node 1111, is changed from 7 to 5. (d) Spare node 1100 recovers the loss of state 7, completing the reconfiguration.

With the definition of the cost for reconfiguring an embedding due to a fault, we have the following definition for a *1-step recoverable embedding*: An embedding is 1-step recoverable if any single fault occurring the embedding can be reconfigured to maintain a fault-free structure of the ring embedded by changing at most one state in a fault-free node.

#### III. An Efficient Scheme

In this section, we present an efficient scheme to systematically construct a 1SRE of a length- $k$  (even) ring in a  $d$ -cube, where  $6 \leq k \leq (3/4)2^d$  and  $d \geq 3$ . Our scheme is based on the following idea: if there exists a 1SRE of a length- $k_1$  ring in a  $d$ -cube and a 1SRE of a length- $k_2$  ring in a  $d$ -cube, then there should exist a 1SRE of a length- $(k_1 + k_2)$  ring in a  $(d+1)$ -cube.

An embedding of a length- $k$  ring in a  $d$ -cube is specified completely by listing the  $k$  active nodes  $a_1, a_2, \dots, a_k$  in order. For example, on the 3-cube, a listing might be

000, 001, 011, 111, 110, 100.

Ignoring the starting node, a more compact notation is to list in order only the coordinate places in which the change occurs. In the example cited, one would obtain  $\langle 1, 2, 3, 1, 2, 3 \rangle$ . This  $k$ -tuple of coordinate places will be called the *transition sequence* for the embedding. In this paper, we will use the notation,  $R_k \rightarrow Q_d$ , to denote an embedding of a length- $k$  ring in a  $d$ -cube. We also define  $R_k \rightarrow Q_d \equiv S \mid T$  where  $S$  is the binary label of the starting node (0 for 00...0) and  $T$  is the transition sequence of the embedding. For example, the embedding of a length-10 ring in a 4-cube in Fig. 2(a) is referred as  $R_{10} \rightarrow Q_4 \equiv 0 \mid \langle 1, 2, 3, 1, 4, 1, 2, 3, 1, 4 \rangle$ .

**Lemma 3.1:** For odd length rings and rings with length less than 6, there are no 1SRE's in the hypercube.

*Proof:* Since there are no odd length cycles as subgraphs of the hypercube [11], there are no embeddings of odd length rings in the hypercube. It is easy to see why there is no 1SRE of length-2 and length-4 rings in the hypercube.  $\square$

**Lemma 3.2:** If there exists a 1SRE of a length- $k$  ring in a  $d$ -

cube, then there exists a 1SRE of a length- $k$  ring in a  $(d+1)$ -cube.

*Proof:* Just leave one  $d$ -cube unused, then the same 1SRE of the length- $k$  ring in a  $d$ -cube is also a 1SRE for a length- $k$  ring in a  $(d+1)$ -cube.  $\square$

**Lemma 3.3:** If there exists a 1SRE of a length- $k_1$  ring in a  $d$ -cube and a 1SRE of a length- $k_2$  ring in a  $d$ -cube, then there exists a 1SRE of a length- $(k_1+k_2)$  ring in a  $(d+1)$ -cube.

*Proof:* First, we present a compositional method to form a 1SRE of a length- $(k_1+k_2)$  ring in a  $(d+1)$ -cube from two existing 1SRE's of length- $k_1$  and length- $k_2$  rings in a  $d$ -cube. Let the two existing 1SRE's be

$$\begin{aligned} R_{k_1} \rightarrow Q_d &\equiv 0 | < x_1, y_1, z_1, \dots >, \text{ and} \\ R_{k_2} \rightarrow Q_d &\equiv 0 | < x_2, y_2, z_2, \dots >. \end{aligned}$$

By swapping some coordinates in the transition sequence of  $R_{k_1} \rightarrow Q_d$ , we can have the first three coordinate places of both transition sequences be the same. For example, if  $R_{k_1} \rightarrow Q_d \equiv 0 | < 1, 2, 3, 1, 2, 4, \dots >$  and  $R_{k_2} \rightarrow Q_d \equiv 0 | < 1, 4, 3, \dots >$ , then we have  $R_{k_1} \rightarrow Q_d \equiv 0 | < 1, 4, 3, 1, 4, 2, \dots >$  after the swapping (exchange 2 and 4). Then, we change the starting node of the new  $R_{k_1} \rightarrow Q_d$  to be  $0 \oplus 2^{y_2-1}$  ( $\oplus$ : the bitwise-XOR operation). So, we have

$$\begin{aligned} R_{k_1} \rightarrow Q_d &\equiv 0 \oplus 2^{y_2-1} | < x_2, y_2, z_2, \dots >, \text{ and} \\ R_{k_2} \rightarrow Q_d &\equiv 0 | < x_2, y_2, z_2, \dots > \end{aligned}$$

such that both embeddings remain 1-step recoverable.

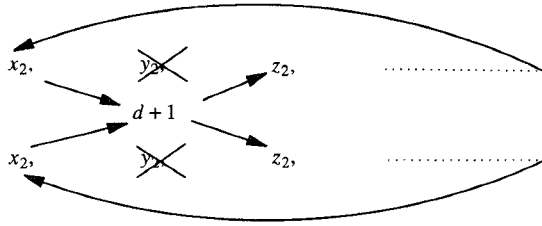


Fig. 3. A transition sequence for  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$  by composing two transition sequences of  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$ .

To form a transition sequence for a 1SRE  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$ , we simply replace the first  $y_2$  coordinate place in both transition sequences for  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$  with  $d+1$  coordinate place, and then combine these two transition sequences together. This processing is depicted in Fig. 3. Thus, we have

$$R_{k_1+k_2} \rightarrow Q_{d+1} \equiv 0 | < x_2, d+1, z_2, \dots, x_2, d+1, z_2, \dots >$$

Second, we need to show that the embedding  $R_{k_1+k_2} \rightarrow Q_{d+1}$  cited is a 1SRE. Fig. 4 will help us to explain this proof more clearly. Fig. 4(a) and 4(b) depict the partial portions of  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$ , respectively. Fig. 4(c) depicts the  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$ , constructed by following the composition method. Note that, nodes  $s_1$  and  $s_2$  in Fig. 4(a) and nodes  $s_3$  and  $s_4$  in Fig. 4(b) must be spare nodes. For  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$  in Fig. 4(c), all the active nodes except nodes  $a_{i_1}$ ,  $a_{i_2}$ ,  $a_{i_3}$ , and  $a_{i_4}$  remain to be 1-step recoverable as they are in  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$ . And, all the spare nodes in  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$  remain to be spare nodes as they join in  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$ . Thus, we need only

to check the recovery processes for nodes  $a_{i_1}$ ,  $a_{i_2}$ ,  $a_{i_3}$ , and  $a_{i_4}$  for verifying the 1-step recoverability for  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$ . Since active nodes  $a_{i_1}$ ,  $a_{i_2}$ ,  $a_{i_3}$ , and  $a_{i_4}$  can be recovered by spare nodes  $s_3$ ,  $s_4$ ,  $s_1$ , and  $s_2$  in one step, respectively, this  $R_{(k_1+k_2)} \rightarrow Q_{(d+1)}$  is a 1SRE.  $\square$

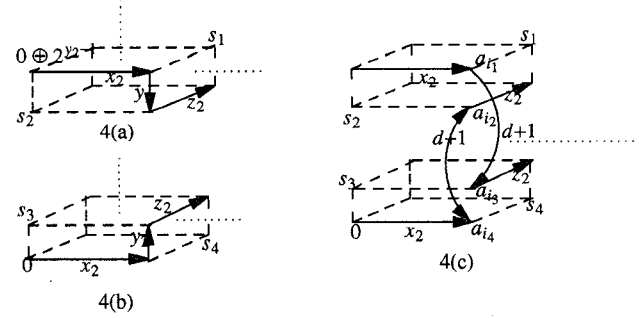


Fig. 4. (a) Partial portion of  $R_{k_1} \rightarrow Q_d \equiv 0 \oplus 2^{y_2-1} | < x_2, y_2, z_2, \dots >$ . (b) Partial portion of  $R_{k_2} \rightarrow Q_d \equiv 0 | < x_2, y_2, z_2, \dots >$ . (c)  $R_{k_1+k_2} \rightarrow Q_{d+1}$  composed from  $R_{k_1} \rightarrow Q_d$  and  $R_{k_2} \rightarrow Q_d$ .

**Example:** By using the composition method in Lemma 3.3, we demonstrate the construction of a 1SRE of a length-20 ring in a 5-cube by two smaller 1SRE's,  $R_8 \rightarrow Q_4 \equiv 0 | < 1, 2, 3, 4, 1, 2, 3, 4 >$  and  $R_{12} \rightarrow Q_4 \equiv 0 | < 1, 4, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ . With the changing of starting node and swapping the elements in the transition sequence, we have  $R_8 \rightarrow Q_4 \equiv 0 \oplus 2^3 | < 1, 4, 3, 2, 1, 4, 3, 2 >$  and  $R_{12} \rightarrow Q_4 \equiv 0 | < 1, 4, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ . Deleting two coordinate-4 places and combining the two sequences together by adding two coordinate-5 places, we have  $R_{20} \rightarrow Q_5 \equiv 0 | < 1, 5, 3, 2, 1, 4, 3, 2, 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ , which is a 1SRE. Fig. 5 depicts such construction of a 1SRE.

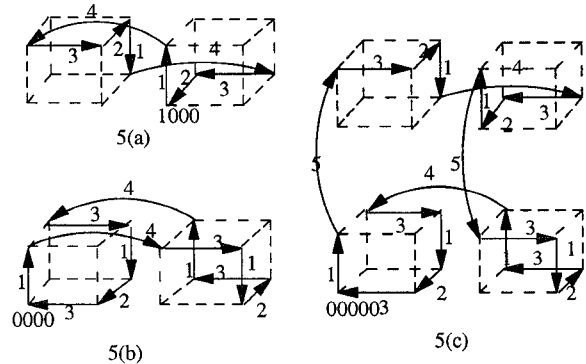


Fig. 5.

- (a) 1SRE  $R_8 \rightarrow Q_4 \equiv 0 \oplus 2^3 | < 1, 4, 3, 2, 1, 4, 3, 2 >$ .
- (b) 1SRE  $R_{12} \rightarrow Q_4 \equiv 0 | < 1, 4, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ .
- (c) 1SRE  $R_{20} \rightarrow Q_5 \equiv 0 | < 1, 5, 3, 2, 1, 4, 3, 2, 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ .

**Lemma 3.4:** For a length-6 ring, there is a 1SRE in a 3-cube.

*Proof:*  $R_6 \rightarrow Q_3 \equiv 0 | < 1, 2, 3, 1, 2, 3 >$ , is a 1SRE.  $\square$

**Lemma 3.5:** For length-6, 8, and 12 rings, there are 1SRE's in a 4-cube.

*Proof:* The proof is divided into following cases:

**Case I.** For a length-6 ring,  $R_6 \rightarrow Q_4 \equiv 0 | < 1, 2, 3, 1, 2, 3 >$  is a 1SRE (Lemma 3.2 and Lemma 3.4).

**Case II.** For a length-8 ring,  $R_8 \rightarrow Q_4 \equiv 0 | < 1, 2, 3, 4, 1, 2, 3, 4 >$  is a 1SRE.

**Case III.** For a length-12 ring, by using the composition method in Lemma 3.3, we can construct a 1SRE by combining two 1SRE's of  $R_6 \rightarrow Q_3 \equiv 0 | < 1, 2, 3, 1, 2, 3 >$ . We have  $R_{12} \rightarrow Q_4 \equiv 0 | < 1, 4, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ , which is a 1SRE. These 1SRE's are shown in Table I.

**Case IV.** For a length-10 ring, an easy computer program was used to enumerate all possibilities, and we found out there is no 1SRE for a length-10 ring in a 4-cube.  $\square$

TABLE I. 1SRE's of length-14, 16, 18, 20, and 24 rings in a 5-cube

Embeddings
$R_{14} \rightarrow Q_5 \equiv 0   < 1, 5, 3, 4, 1, 2, 3, 4, 1, 5, 3, 1, 2, 3 >$ Constructed From: $R_6 \rightarrow Q_4, R_8 \rightarrow Q_4$
$R_{16} \rightarrow Q_5 \equiv 0   < 1, 5, 3, 4, 1, 2, 3, 4, 1, 5, 3, 4, 1, 2, 3, 4 >$ Constructed From: $R_8 \rightarrow Q_4, R_8 \rightarrow Q_4$
$R_{18} \rightarrow Q_5 \equiv 0   < 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3, 1, 5, 3, 1, 4, 3 >$ Constructed From: $R_6 \rightarrow Q_4, R_{12} \rightarrow Q_4$
$R_{20} \rightarrow Q_5 \equiv 0   < 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3, 1, 5, 3, 2, 1, 4, 3, 2 >$ Constructed From: $R_8 \rightarrow Q_4, R_{12} \rightarrow Q_4$
$R_{24} \rightarrow Q_5 \equiv 0   < 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3, 1, 5, 3, 1, 2, 3, 1, 4, 3, 1, 2, 3 >$ Constructed From: $R_{12} \rightarrow Q_4, R_{12} \rightarrow Q_4$

**Lemma 3.6:** For length-6, 8, 10,  $\dots$ , 18, 20, 24 rings, there are 1SRE's in a 5-cube, but no 1SRE of a length-22 ring in a 5-cube.

*Proof:* The proof is divided into the following cases:

**Case I.** For length-6, 8, and 12 rings, there are 1SRE's in a 5-cube (Lemma 3.2 and Lemma 3.5).

**Case II.** For a length-10 ring, the embedding  $R_{10} \rightarrow Q_5 \equiv 0 | < 1, 2, 3, 4, 5, 1, 2, 3, 4, 5 >$  is a 1SRE.

**Case III.** For length-14, 16, 18, 20, and 24 rings, there are 1SRE's in a 5-cube (Lemma 3.3).

**Case IV.** For a length-22 ring, the composition method does not work, since  $22 = 12 + 10$  and there is no 1SRE of a length-10 ring in a 4-cube. A computer program was used to enumerate all possibilities, and it shows there is no 1SRE of a length-22 ring in a 5-cube.

**Lemma 3.7:** For length-6, 8, 10,  $\dots$ , 44, 46, 48 rings (i.e., rings of length  $k$ ,  $6 \leq k \leq (3/4)2^6$ ), there are 1SRE's in a 6-cube.

*Proof:* The proof is divided into the following cases:

**Case I.** For length-6, 8,  $\dots$ , 20, and 24 rings, there are 1SRE's in a 6-cube (Lemma 3.2 and Lemma 3.6).

**Case II.** For length-22, 26,  $\dots$ , 44, and 48 rings, there are 1SRE's in a 6-cube (Lemma 3.3). A summary of constructing these 1SRE's is given in Table II.

**Case III.** For a length-46 ring, the composition method does not work, since  $46 = 24 + 22$  and there is no 1SRE for a length-22 ring in a 5-cube. However, the following embedding comes out,

$$R_{46} \rightarrow Q_6 \equiv 0 | < 1, 2, 3, 1, 2, 4, 1, 2, 3, 1, 2, 5, 1, 2, 3, 1, 2, 4, 1, 2, 3, 1, 6, 3, 4, 1, 2, 4, 1, 5, 4, 3, 5, 1, 2, 5, 1, 3, 4, 1, 3, 2, 5, 1, 6, 5 >$$

which is a 1SRE.  $\square$

TABLE II. Summary of constructing 1SRE's of length-22, 26,  $\dots$ , 44 and 48 rings in a 6-cube

Embeddings	Constructed from	Embeddings	Constructed from
$R_{22} \rightarrow Q_6$	$R_{10} \rightarrow Q_5, R_{12} \rightarrow Q_5$	$R_{36} \rightarrow Q_6$	$R_{18} \rightarrow Q_5, R_{18} \rightarrow Q_5$
$R_{26} \rightarrow Q_6$	$R_{12} \rightarrow Q_5, R_{14} \rightarrow Q_5$	$R_{38} \rightarrow Q_6$	$R_{18} \rightarrow Q_5, R_{20} \rightarrow Q_5$
$R_{28} \rightarrow Q_6$	$R_{14} \rightarrow Q_5, R_{14} \rightarrow Q_5$	$R_{40} \rightarrow Q_6$	$R_{20} \rightarrow Q_5, R_{20} \rightarrow Q_5$
$R_{30} \rightarrow Q_6$	$R_{14} \rightarrow Q_5, R_{16} \rightarrow Q_5$	$R_{42} \rightarrow Q_6$	$R_{18} \rightarrow Q_5, R_{24} \rightarrow Q_5$
$R_{32} \rightarrow Q_6$	$R_{16} \rightarrow Q_5, R_{16} \rightarrow Q_5$	$R_{44} \rightarrow Q_6$	$R_{20} \rightarrow Q_5, R_{24} \rightarrow Q_5$
$R_{34} \rightarrow Q_6$	$R_{16} \rightarrow Q_5, R_{18} \rightarrow Q_5$	$R_{48} \rightarrow Q_6$	$R_{24} \rightarrow Q_5, R_{24} \rightarrow Q_5$

**Theorem 3.1:** There exists a 1SRE of a length- $k$  (even) ring in a  $d$ -cube, where  $6 \leq k \leq (3/4)2^d$  and  $d \geq 3$ , except when  $k=10$  with  $d=4$ , and  $k=22$  with  $d=5$ .

*Proof:* We prove this theorem by an induction proof based on  $d$ .

**Induction base:** For  $d = 3$ ,  $d = 4$ ,  $d = 5$ , and  $d = 6$ , the theorem follows based on Lemma 3.4, Lemma 3.5, Lemma 3.6, and Lemma 3.7, respectively.

**Induction hypothesis:** Assume the theorem follows when  $d = m \geq 6$ .

**Induction step:**  $d = m + 1$

**Case I:**  $6 \leq k \leq (3/4)2^m$ . According to induction hypothesis and Lemma 3.2, there exists a 1SRE of a  $k$  length ring in a  $(m+1)$ -cube.

**Case II:**  $(3/4)2^m < k \leq (3/4)2^{(m+1)}$ . Since  $k$  is even, we have  $k = k_1 + k_2$ , where  $k_1$  and  $k_2$  are even, and  $6 \leq k_1 \leq (3/4)2^m$  and  $6 \leq k_2 \leq (3/4)2^m$ . By the induction hypothesis there exists a 1SRE of a  $k_1$  length in a  $m$ -cube and a 1SRE of a  $k_2$  length in a  $m$ -cube. Using the composition method in Lemma 3.3, we can construct a 1SRE of a length- $k$  ring in a  $(m+1)$ -cube with these two smaller 1SRE's.  $\square$

## IV. Complexity and Comparison

The running time complexity of our schemes for embedding a length- $n$  ring in the hypercube is approximately about

$$T(n) = 2T(n/2) + c,$$

and we have  $T(n) = O(n)$ . So, the running time complexity of our scheme is polynomial to the length of the ring embedded.

The comparison of our scheme to the existing schemes for 1SRE's (i.e., [5,8,11]) is described as follows. Basically, the lengths of rings that are applicable to other schemes in [5,11] are very restricted, since both schemes are designed for embedding rings of lengths  $2^{d-1}$  and  $(3/4)2^d$  only in a  $d$ -cube. It is shown that for rings of such lengths both schemes have efficient ways to construct 1SRE's, however, they are not able to construct 1SRE's of rings of other lengths. Compared with the schemes in [5,11], our scheme can be applied to more rings of different lengths.

Unlike the schemes in [5,11], the scheme in [8] can construct 1SRE's of any even length ring in the hypercube. However, the trade-off for such excellent performance is that the resulting 1SRE's are of large expansion (the ratio of the size (in number of nodes) of the embedding hypercube to that of the embedded ring). For many case, the scheme will embed a ring in a hypercube that is one dimension larger than is necessary in order to achieve efficient reconfiguration due to faults. For example, a length-16 or length-20 ring will be embedded in a 6-cube by the scheme, although a 5-cube is enough to accommodate the ring. Compared with the scheme in [8], our scheme has better performance in terms of the processor utilization of the resulting 1SRE's.

We summarize the comparison of our scheme to other existing schemes for constructing 1SRE's in Table III.

TABLE III. Comparison among schemes for constructing 1SRE's

Schemes	Applicability to the ring	Expansion of resulting 1SRE's
[5]	Rings of length $2^{d-1}$ and $(3/4)2^d$ in a $d$ -cube	-
[11]	Rings of length $(3/4)2^d$ in a $d$ -cube	-
[8]	All even length rings	Large expansion in many cases
Our scheme	Rings of length from 6 to $(3/4)2^d$ in a $d$ -cube	Smaller expansion than that for [8]

### V. A Sufficient Condition for the Non-existence of 1SRE's

In this section, we discuss a sufficient condition for the non-existence of a 1SRE of a length- $k$  (even) ring in a  $d$ -cube where  $k > (3/4)2^d$  and  $d \geq 3$ . The sufficient condition is based on the idea: no spare node can serve more than  $d - 1$  active nodes when  $k > (3/4)2^d$ .

**Definition :** Let  $a_i, a_{i+1}$ , and  $a_{i+2}$  be three hypercube nodes with consecutive active states on a 1SRE, and  $s$  be the label of a spare node where  $s = \text{XOR}(a_i, a_{i+1}, a_{i+2})$ .

- (1) The spare node  $s$  will be referred as the *1-step recovery spare node* for active node  $a_{i+1}$  in the 1SRE.
- (2) Nodes  $s, a_i, a_{i+1}$ , and  $a_{i+2}$  will constitute the four nodes of a 2-D plane of the hypercube. This 2-D plane will be referred as a *1-step recovery hyperplane* (1SRHP) with respect to  $s$ , and denoted specifically as  $[s, a_i, a_{i+1}, a_{i+2}]$

**Lemma 5.1:** For a 1SRE, it is impossible to have more than two 1SRHP's (with respect to the same spare node) that intersect on a hypercube link with the spare node as one endpoint.

*Proof :* Suppose there exists a valid 1SRE where three 1SRHP's with respect to the same spare node intersect on a link with the spare node as one endpoint. Fig. 6 shows such a situation where three 1SRHP's,  $[s, a_{i1}, a_{i2}, a_{i3}]$ ,  $[s, a_{i3}, a_{i6}, a_{i7}]$ , and  $[s, a_{i3}, a_{i4}, a_{i5}]$ , intersecting on the hypercube link  $(s, a_{i3})$ . By definition of 1SRHP, the three hypercube links,  $(a_{i2}, a_{i3})$ ,  $(a_{i3}, a_{i4})$ , and  $(a_{i3}, a_{i6})$ , must all be the image links of the embedding. This is a contradiction.  $\square$

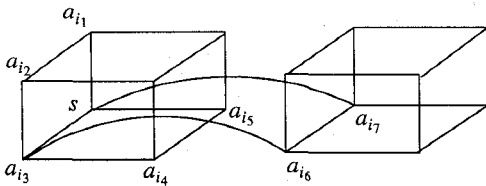


Fig. 6. Three 1SRHP's with respect to  $s$  intersect on the edge  $(s, a_{i3})$ .

**Theorem 5.1:** For a 1SRE in a  $d$ -cube, each spare node can serve as a 1-step recovery spare node for at most  $d$  active nodes.

*Proof :* According to Lemma 5.1, the best recovery case for a spare node is that each hypercube link incident to it is contained in exactly two 1SRHP's with respect to it. Such a situation is depicted in Fig. 7.  $\square$

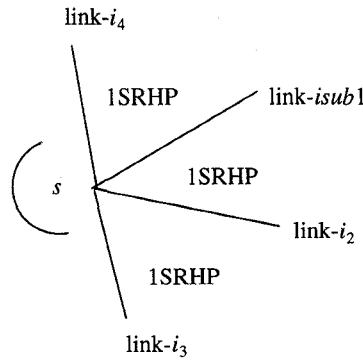


Fig. 7. Each hypercube link incident to the spare node  $s$  is contained in exactly two 1SRHP's with respect to  $s$ .

**Lemma 5.2:** If  $[s, a_{i1}, a_{i2}, a_{i3}]$  and  $[s, a_{i3}, a_{i4}, a_{i5}]$  are two 1SRHP's with respect to spare node  $s$  in a 1SRE, then  $a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}$  or  $a_{i5}, a_{i4}, a_{i3}, a_{i2}, a_{i1}$  must be contained in the active node listing (in order) of the 1SRE.

*Proof :* According to the definition of 1SRHP,  $[s, a_{i1}, a_{i2}, a_{i3}]$  is a 1SRHP with respect to  $s$  if  $a_{i1}, a_{i2}, a_{i3}$  or  $a_{i3}, a_{i2}, a_{i1}$  is contained in the active node listing of the 1SRE. Similarly,  $a_{i3}, a_{i4}, a_{i5}$  or  $a_{i5}, a_{i4}, a_{i3}$  must also be contained in the active node listing of the 1SRE. Thus, by combination,  $a_{i1}, a_{i2}, \dots, a_{i5}$  or  $a_{i5}, a_{i4}, \dots, a_{i1}$  must be contained in the active node listing of the 1SRE.  $\square$

**Lemma 5.3:** For a 1SRE of a length- $k$  ring in a  $d$ -cube, if a spare node serves for  $d$  active nodes as a 1-step recovery spare node, then  $k = 2d$ .

*Proof :* From Fig. 7, we can see that if a spare node serves for  $d$  active nodes as 1-step recovery spare node, then each hypercube link incident to this spare node must be contained in exactly two 1SRHP's with respect to this spare node. Such a situation is depicted in Fig. 8 where  $s$  is a spare node,  $n_i$  (in active state) are neighbors to  $s$ , and  $a_i$  are active nodes which use  $s$  as 1-step recovery spare node. Then, according to Lemma 5.2,

$$\begin{aligned} & n_1, a_1, n_2, a_2, n_3, \\ & n_2, a_2, n_3, a_3, n_4, \\ & \dots \\ & n_{d-1}, a_{d-1}, n_d, a_d, n_1 \end{aligned}$$

must be all contained in the active node listing in order of the 1SRE. Since the combination of these sublists forms a cycle of active nodes (i.e.,  $n_1, a_1, \dots, n_d, a_d, n_1$ ), the length of this cycle must be equal to the length of the ring embedded. That is,  $k = 2d$ .  $\square$

**Theorem 5.2:** For a 1SRE of a length- $k$  ring in a  $d$ -cube where  $k > (3/4)2^d$ , each spare node can serve as a 1-step recovery spare node for at most  $d - 1$  active nodes.

*Proof :* Since  $(3/4)2^d \geq 2d$  for  $d \geq 3$ , according to Theorem 5.1 and Lemma 5.3 to embed a length- $k$  ring in a  $d$ -cube as a 1SRE where  $k > (3/4)2^d$ , a spare node can serve for at most  $d - 1$  active nodes as 1-step recovery spare node.  $\square$

**Theorem 5.3:** There is no 1SRE of a length- $((3/4)2^d + q)$  (even) ring in a  $d$ -cube, if  $\frac{2^d}{4} - \frac{2^d}{d} < q$ .

*Proof :* As a consequence of Theorem 5.2, if the number of the total number of spare nodes multiplying  $d - 1$  is less than the

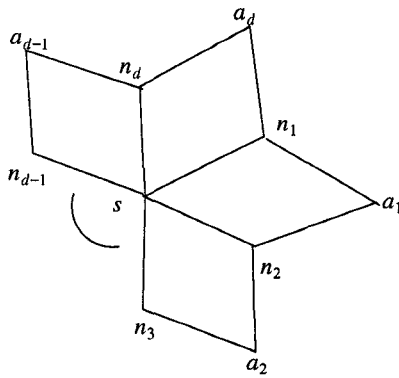


Fig. 8. A spare node  $s$  is used for  $d$  active nodes as 1-step recovery spare node.

total number of active nodes, then some active nodes will not be able to be recovered in one step. That is, if the following inequality is satisfied,

$$\begin{aligned} & \left(\frac{1}{4} 2^d - q\right) \times (d-1) < \frac{3}{4} 2^d + q \\ \Rightarrow & \frac{d}{4} 2^d - \frac{1}{4} 2^d - qd + q < \frac{3}{4} 2^d + q \\ \Rightarrow & \frac{2^d}{4} - \frac{2^d}{d} < q, \end{aligned}$$

the embedding is not a 1SRE.  $\square$

By Theorem 5.3, we have the summary of those lengths for rings having no 1SRE's in a  $d$ -cube in Table IV.

TABLE IV. Rings of length  $> (3/4)2^d$  and  $< 2^d$  having no 1SRE's in a  $d$ -cube

$d$	$(3/4)2^d$	$\frac{2^d}{4} - \frac{2^d}{d}$	Lengths of rings having no 1SRE's in a $d$ -cube
4	12	0	14
5	24	1.6	26, 28, 30
6	48	5.3	54, 56, 58, 60, 62
7	96	13.9	110, 112, 114, ..., 126
...			

## VI. Concluding Remarks and Open Problems

We have presented a scheme to systematically construct a 1SRE of a length- $k$  (even) ring in a  $d$ -cube, where  $6 \leq k \leq (3/4)2^d$  and  $d \geq 3$ . Results show that our scheme has better performance, in terms of applicability to rings and expansion of the resulting embeddings, than that of other schemes.

A sufficient condition for the non-existence of 1SRE's for rings of length  $> (3/4)2^d$  in a  $d$ -cube is also addressed in this paper. Clearly there exists a gap between  $(3/4)2^d$  and the lower bound of lengths for rings having no 1SRE's in a  $d$ -cube,  $d \geq 6$  (Table IV), and the size of the gap increases as  $d$  increases. It remains an open problem to diminish the size of the gap by either improving the lower bound for non-existence of 1SRE's or extending the  $(3/4)2^d$  bound for the existence of 1SRE's. Experimentally, we have determined there exist 1SRE's of rings of length  $> (3/4)2^d$  in large  $d$ -cubes.

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